NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 4

Spring 2025

Suppose $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are the roots of the degree 5 polynomial

$$f(x) = x^5 - 2x^4 - 6x^3 + 6x^2 + 6x + 1.$$

Then find the value of $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2$. Justify your answer.

Solution. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are the roots $f(x)$, then $f(x)$ must equal

$$f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)(x - \alpha_5).$$

Multiplying out the factors of the above expression, we notice that coefficient of x^4 is the sum $-(\alpha_1 + \cdots + \alpha_5)$. Thus, we have

$$\sum_{1 \le i \le 5} \alpha_i = 2.$$

Likewise, comparing coefficients of x^3, we get

$$\sum_{1 \le i < j \le 5} \alpha_i \alpha_j = -6.$$

Therefore,

$$\begin{aligned} \sum_{1 \le i \le 5} \alpha_i^2 &= \left(\sum_{1 \le i \le 5} \alpha_i \right)^2 - 2 \left(\sum_{1 \le i < j \le 5} \alpha_i \alpha_j \right) \\ &= 2^2 - 2(-6) \\ &= 16. \end{aligned}$$